Integrability in AdS/CFT

Benjamin BASSO

Laboratoire de Physique Théorique d'Orsay
Université de Paris XI

October 31, 2008
SCSC08

Outline

- Integrability on gauge theory side
- Integrability on string theory side
- Merging of gauge and string integrable structures
- Outlook

$\mathcal{N} = 4$ Super-Yang-Mills Theory

Action

$$\mathcal{S} = -\frac{1}{2g_{\rm YM}^2}\int d^4x\,{\rm Tr}\big(F_{\mu\nu}F^{\mu\nu}\big) + \text{fermions} + \text{scalars}$$

with all fields in the adjoint representation of the gauge group SU(N_c)

Classical symmetries

- Conformal invariance $\longrightarrow so(4,2)$
- $\mathcal{N} = 4$ supersymmetries $\longrightarrow su(4) \sim so(6)$ (R-symmetry)

They combine into the superconformal invariance $psu(2,2|4)$

Quantum symmetries

The coupling constant does not run ($\beta(g_{\rm YM}^2) = 0$) *and the superconformal invariance is preserved at the quantum level*

Hidden quantum symmetries in the 't Hooft planar limit? ($N_c = \infty$ and $g^2 = g_{\rm YM}^2 N_c/16\pi^2$ fixed)

Planar dilatation operator of the conformal theory is expected to be completely integrable!

- Definitely established at one-loop in pertubation theory
- Supported by higher-loop investigations

Type IIB String Theory on $AdS_5 \times S^5$

World-sheet action

$$\mathcal{S} = \frac{T}{2} \int d^2\sigma \sqrt{h}\, h^{\alpha\beta}\, G_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu + \text{fermions}$$

- Background $AdS_5 \times S^5$ geometry encoded into $ds^2 = G_{\mu\nu}(X) dX^\mu dX^\nu$
- String tension $T\,(= \frac{1}{2\pi\alpha'}) = 2g$

When the coupling gets large, quantum corrections are suppressed and the string sigma model becomes tractable perturbatively in $1/g$

Integrability of the classical string sigma model!

Classical string solutions $\longleftrightarrow$ highly-excited states carrying large charges (angular momentum, ...)

AdS/CFT Correspondence

Maldacena conjecture

(Planar) $\mathcal{N} = 4$ SU(N_c) Super-Yang-Mills is dual to type IIB (non-interacting) string theory on $AdS_5 \times S^5$ background geometry

$$\text{string tension} \qquad T = \frac{\sqrt{g_{\rm YM}^2 N_c}}{2\pi} = 2g \tag{1}$$

$$\text{string coupling} \qquad g_s \propto \frac{1}{N_c} \tag{2}$$

Mapping between conformal operators and string states

Conformal operator with global quantum numbers $S, L, \ldots$ and scaling dimension $\Delta_{S,L}(g)$	$\longleftrightarrow$	String state with conserved charges $S, L, \ldots$ and energy $E_{S,L}(g) = \Delta_{S,L}(g)$

Motivations

- Prove/Disprove the AdS/CFT correspondence (at least in the planar limit) by comparing gauge and string integrable structures (once they have been identified)
- Explore and develop an alternative (and efficient) quantization of string theory on certain curved backgrounds
- Explore secret (and fully quantum) symmetries of gauge theories
- Identify new completely integrable systems

Anomalous Dimensions of Conformal Operators

sl(2) sector

Single-trace operators built from L complex scalars $\mathcal{Z}(0)$ and S lightcone derivatives D_+

$$\mathcal{O}_{\mathbf{n}}(0) = \mathrm{Tr}\{D_+^{n_1}\mathcal{Z}(0)...D_+^{n_L}\mathcal{Z}(0)\} \qquad \mathbf{n} = (n_1, ..., n_L) \in \mathbb{N}^L$$

Quantum numbers : Twist L and Lorentz spin $S = n_1 + ... + n_L$

Dilatation operator :

Operators with same quantum numbers mix under a change of the renormalization scale according to the Callan-Symanzik equation

$$\mu\frac{\partial}{\partial\mu}\mathcal{O}_{\mathbf{n}}(0) = (\mathbb{H}\cdot\mathcal{O})_{\mathbf{n}}(0) \qquad \mathbb{H} = \text{dilatation operator}$$

Operators $\mathcal{O}_{\mathbf{n}}(0)$ belong to a closed $sl(2)$ sector

Anomalous dimension

Spectrum of the dilatation operator = anomalous dimensions of conformal operators

$$\delta_{L,S}(g) = \Delta_{L,S}(g) - S - L \qquad \delta_{L,S}(g) = \mathcal{O}(g^2)$$

Scaling dimension $\Delta_{L,S}(g)$

$$\langle\mathcal{O}(x)\mathcal{O}(y)\rangle \sim \left[(x-y)^2\right]^{-\Delta_L(S)}$$

Spin Chain Representation

Kinematics

Single-trace composite operators built from L complex scalar fields $\mathcal{Z}(0)$ and S lightcone derivatives D_+

$$\mathcal{O}_{\mathbf{n}}(0) = \mathrm{Tr}\{D_+^{n_1}\mathcal{Z}(0)...D_+^{n_L}\mathcal{Z}(0)\} \qquad \mathbf{n} = (n_1, ..., n_L) \in \mathbb{N}^L$$

- $\mathrm{Tr}\{\mathcal{Z}(0)...\mathcal{Z}(0)...\mathcal{Z}(0)\} \rightarrow$ vacuum state of the spin chain
- $\mathrm{Tr}\{\mathcal{Z}(0)...D_+\mathcal{Z}(0)...\mathcal{Z}(0)\} \rightarrow$ one-particle state of the spin chain (magnon)

Quantum numbers :

- Twist $L \rightarrow$ spin chain length
- Lorentz spin $S = n_1 + ... + n_L \rightarrow$ number of excitations (magnons) over the vacuum

Dynamics

Operators with same quantum numbers mix under a change of the renormalization scale according to the Callan-Symanzik equation

$$\mu\frac{\partial}{\partial\mu}\mathcal{O}_{\mathbf{n}}(0) = (\mathbb{H}\cdot\mathcal{O})_{\mathbf{n}}(0)$$

- $\mathbb{H}$ dilatation operator of the conformal $\mathcal{N} = 4$ gauge theory $\rightarrow$Hamiltonian of the spin chain
- Spectrum of anomalous dimensions $\delta_{L,S}(g) \rightarrow$ spectrum of energies of the spin chain

$\mathbb{H}$ is integrable and can be solved by means of Bethe ansatz

Bethe ansatz for (planar) dilatation operator (one-loop example)

- Bethe equations for integrable $sl(2)$ Heisenberg spin chain

$$\left(\frac{u_k + \imath/2}{u_k - \imath/2}\right)^L = \prod_{j\neq k}^{S} \frac{u_k - u_j - \imath}{u_k - u_j + \imath} \quad \text{(quantization conditions)}$$

- One-loop anomalous dimension

$$\delta_{L,S}(g) = g^2 \sum_{k=1}^{S} \frac{1}{u_k^2 + 1/4} \quad \text{(energy)}$$

- Extra constraint on physical state

$$\prod_{k=1}^{S} \frac{u_k + \imath/2}{u_k - \imath/2} = 1 \quad \text{(cyclicity or zero-momentum condition)}$$

Dual String Picture

String states

String rotating in AdS_3 with angular momentum S and boosted along a big circle of S_5 with angular momentum L

For large conserved charges and in the limit of large tension, these states can be analysed in the classical approximation

$$S, L \sim g \qquad \text{with} \qquad g \to \infty \qquad \longleftrightarrow \qquad \text{classical strings}$$

String evolution

Classical (string) equations of motion are complicated non-linear partial differential equations

Integrability of these equations allows to solve them in terms of algebraic curves equipped with a meromorphic differential

These curves can be interpreted as the continuum-limit (large number of excitations) description of some Bethe ansatz equations

Discretize the algebraic curve and unify the gauge and string integrable structures!

Asymptotic S-matrix

Program

- Spectrum of asymptotic elementary excitations
- Diffractionless scattering
- Factorized scattering
- Crossing symmetry

All-loop proposal for the two-body S-matrix (sl(2) sector)

$$S(u_k, u_j) = \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \frac{1 - g^2/x_k^+ x_j^-}{1 - g^2/x_k^- x_j^+} \exp\left(2\imath\theta(u_k, u_j)\right)$$

- Deformed spectral parameter $u_k \pm \imath/2 = x_k^\pm + g^2/x_k^\pm$
- Dressing phase $\theta(u_k, u_j)$ whose explicit expression was recently guessed

All-loop energy and impulsion of an asymptotic elementary excitation

$$E_k = g^2\left(\frac{\imath}{x_k^+} - \frac{\imath}{x_k^-}\right) \qquad e^{iP_k} = \left(\frac{x_k^+}{x_k^-}\right)$$

Conclusion and Outlook

- Planar $\mathcal{N} = 4$ super Yang-Mills theory and/or non-interacting type IIB string theory on $AdS_5 \times S^5$ are expected to be completely integrable
- Definitely established at one-loop in the gauge theory and classically in the string theory
- All-loop asymptotic S-matrix for the AdS/CFT sytem has been proposed
- So far its predictions were shown to match gauge and string results with high accuracy

Is it possible to derive the S-matrix from first principles and to prove that indeed it is exactly the same for gauge and string?